STABILITY IN THE MEAN SQUARE OF A SYSTEM OF STOCHASTIC DIFFERENTIAL EQUATIONS

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In this paper we consider objects, whose behavior can be described in terms of solutions of a system of linear differential equations with random coefficients. Each of these solutions represents, in our case, a diffusion process. A system of linear differential equations with constant coefficients is derived for second moments of this diffusion process. This system can, in particular, be used in investigation of mean square stability of a trivial solution of the corresponding system of stochastic differential equations(*).

1. Consider a system of stochastic differential equations of the type

$$dx_i = \sum_{j=1}^{n} a_{ij} x_j \, dt + \sum_{k=1}^{m} \Big(\sum_{j=1}^{n} b_{ij}^{k} x_j \Big) dw_k(t) \tag{1}$$

where $w_k(t)$ ($t = 1,\ldots,m$) are independent Brownian processes.

When initial conditions are given, Eq. (1) defines, as we know [1 to 4], a diffusion process. A stationary Markov transitional function $P(t, x, \Gamma)$ of this process defines a subgroup of linear operatiors T_t

$$T_t f(x) = \int_X P(t, x, dy) f(y) \tag{2}$$

on a Banach space B of all Borel bounded functions $f(x)$ $(x = (x_1,\ldots,x_n))$.

All bounded functions $f(x)$ which have continuous bounded derivatives of first and second order, belong to the domain of definition D_A of the infinitesimal operator A of the subgroup T_t. For such functions, we have the following Formula:

$$Af = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \sum_{k=1}^{m} \Big(\sum_{j=1}^{n} b_{\alpha j}^{k} x_j \sum_{j=1}^{n} b_{\beta j}^{k} x_j \Big) \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} \tag{3}$$

which can be written as

$$Af = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \Big(\sum_{i,j=1}^{n} a_{ij}^{\alpha\beta} x_i x_j \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} \Big) \tag{4}$$

where the coefficients $a_{ij}^{\alpha\beta} = a_{ij}^{\beta\alpha}$ can be found directly from the coefficients b_{ij}^{k}. Integral in the right-hand side of (2) gives mathematical expectation of the function f at the instant t under the condition, that initial distribution of the Markov process given by $P(t, x, \Gamma)$ is concentrated at the point x. This integral may also exist for unbounded functions. For example, it converges for functions of the type $f_{ij}(x) = x_i x_j$. In the following we shall denote the integral of (2) by $T_t f$.

Theorem 1. Functions $T_t f_{kl}(x)$ ($k = 1,\ldots,n$; $l = 1,\ldots,n$) satisfy, at any fixed value

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*) In the proof-reading stage the authors became acquainted with a work by Gikhman[9] in which equations for second moments are obtained by a different method.

of x, a following system of $n(n+1)/2$ linear differential equations with constant coefficients

$$\frac{dg_{kl}}{dt}=\sum_{j=1}^{n} a_{kj}g_{jl}+\sum_{j=1}^{n} a_{lj}g_{jk}+\sum_{i,\,j=1}^{n} a_{ij}^{\ kl}g_{ij} \quad (k\leqslant l) \tag{5}$$

and solution of (5) satisfying the initial values $g_{kl}(0)=x_k x_l$ coincides with the functions $T_t f_{kl}(x)$.

Proof. If $f\in D_A$, then we have the equality [3]

$$dT_t f(x)/dt = AT_t f(x) = T_t Af(x) \tag{6}$$

which, when formally applied to functions f_{kl}, at once yields the system (5). A rigorous proof follows. Let us construct for $f_{kl}(x)$ a sequence of functions $f_{kl}^{N}(x)=y_k^{N}(x)y_l^{N}(x)$ where

$$y_i^N(x)=\begin{cases} x_i, & |x_i|\leqslant N\\ (N+{}^1/_2)\operatorname{sign} x_i, & |x_i|\geqslant N+1\\ {}^1/_2(x_i-N)^4-(x_i-N)^3+x_i, & N<x_i<N+1\\ -{}^1/_2(x_i+N)^4-(x_i+N)^3+x_i, & -N-1<x_i<-N \end{cases} \tag{7}$$

converging to $f_{kl}(x)$. Obviously $f_{kl}^{N}(x)\in D_A$ and the equality [3]

$$T_t f_{kl}^{N}(x)-f(x)=\int_0^t T_s Af_{kl}^{N}(x)\,ds \tag{8}$$

holds. We can easily, at any fixed value of x, pass to the limit in (8) as $N\to\infty$, thus obtaining Eqs. (5), which proves the theorem.

Note 1. Let $\varphi(\Gamma)$ be a finite, completely additive function of Borel sets of the domain X. It is easy to show that functions

$$M_{kl}(t)=(T_t f_{kl}(x),\,\varphi)=\int_X\int_X P(t,\,x,\,dy)\,f_{kl}(y)\,\varphi(dx) \tag{9}$$

(which define the mathematical expectation of functions $f_{kl}(x)$ at the instant t under the condition that $\varphi(\Gamma)$ is the initial distribution of a Markov process with a transitional function $P(t,\,x,\,\Gamma)$) also satisfy (5).

Note 2. In the literature (see e.g. [5]), Eqs. (1) are also interpreted in the sense other than that of Ito [1]. Diffusion process however, resulting from the new interpretation, still possesses an infinitesimal operator of the type given in (4), where the only factors that change in a predictable manner [5] are the constant coefficients of linear forms accompanying first derivatives of the function f. This shows that equations analogous to (5) can be obtained also in this case. Another method of obtaining these equations is given in [6].

2. Trivial solution of (1) shall be called stable in the mean square (see e.g. [7 and 8]) if for any $\varepsilon>0$ such $\delta>0$ can be found that, when $|x|^2=x_1^2+\ldots+x_n^2\leqslant\delta^2$, then $T_t f(x)<\varepsilon$ for all $t>0$ where $f=x_1^2+\ldots+x_n^2$. If in addition $T_t f(x)\to 0$ at $t\to\infty$, then the trivial solution is asymptotically stable in the mean square. Stability of systems of the type (1) was investigated in a number of works (see the Bibliography of [8]). It is clear that for a null solution of (1) to be e.g. asymptotically stable in the mean square, it is necessary and sufficient that it is stable and

$$\lim_{t\to\infty} T_t f_{ij}(x)=0, \qquad \text{for } |x|<\delta \qquad (i=1,\ldots,n;\; j=1,\ldots,n)$$

Theorem 2. Trivial solution of a stochastic system of differential Eqs. (1) is stable (asymptotically stable) in the mean square if and only if the null solution of the system of $n(n+1)/2$ linear differential equations with constant coefficients (5) is stable (asymptotically stable).

Proof. Suppose the trivial solution of a stochastic system of differential Eqs. (1) is stable in the mean square while ε and $\delta=\delta(\varepsilon)$ are numbers appearing in the definition of stability given above. Let $g_{kl}(t)$ ($k=1,\ldots,n$; $l=l,\ldots,n$; $k\leqslant l$) be a solution of (5) with the following initial values

$$g_{k_0k_0}(0) = x_{k_0}^2 \leqslant {}^1/_2\delta^2, \quad g_{l_0l_0}(0) = x_{l_0}^2 \leqslant {}^1/_2\delta^2, \quad g_{k_0l_0}(0) = x_{k_0}x_{l_0}, \quad g_{kl}(0) = 0 \qquad (10)$$

and suppose that $g_{kl}(0) = 0$ for the remaining pairs of indices k and l.

By Theorem 1, $g_{kl}(t) = T_t f_{kl}(x)$. Since $x^2 = x_{k_0}^2 + x_{l_0}^2 \leqslant \delta^2$ therefore when $k = 1, \ldots, n; l = 1, \ldots, n; k \leqslant l$ and $t > 0$,

$$|g_{kl}(t)| = |T_t f_{kl}(x)| \leqslant {}^1/_2(T_t f_{kk}(x) + T_t f_{ll}(x)) \leqslant {}^1/_2 T_t f(x) < {}^1/_2\varepsilon \qquad (11)$$

holds.

We should note that there are $n(n+1)/2$ linearly independent solutions of (5) when initial values are of the type (10). This fact alone is sufficient to prove the stability of a trivial solution of (5). The converse is obvious. The theorem can also be proved for the case of asymptotic stability in the analogous manner.

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